Some theory of bivariate risk attitude*

Marta Cardin and Paola Ferretti

Department of Applied Mathematics
University Ca' Foscari Venice
Dorsoduro 3825/E
I-30123 Venice, Italy
mcardin@unive.it, ferretti@unive.it

Abstract. In past years the study of the impact of risk attitude among risks has become a major topic, in particular in Decision Sciences. Subsequently the attention was devoted to the more general case of bivariate random variables. The first approach to multivariate risk aversion was proposed by de Finetti ([2]) and Richard ([15]) and it is related to the bivariate case. More recently, multivariate risk aversion has been studied by Scarsini ([20], [21], [22]). Nevertheless even if decision problems with consequences described by more than two attributes have become increasingly important, some questions appear not completely solved. This paper concerns with a definition of bivariate risk aversion which is related to a particular type of concordance: a bivariate risk averse Decision Maker is a Decision Maker who always prefers the independent version of a bivariate random variable to the random variable itself.
Keywords. Bivariate risk aversion; concordance aversion; pain functions; submodular functions; bivariate association; concordance; dependence; diversification.

J.E.L. CLASSIFICATION: D810.
M.S.C. CLASSIFICATION: 91B06; 60E15; 62H20.

1 Introduction

As it is well-known, a Decision Maker is risk averse if she always prefers the expected value of a random amount to itself when she has to choice between them. The study of risk attitude has been extensively treated in the framework of Expected Utility Theory generally in the case of unidimensional utility functions, both in presence of one risk both in presence of more risks. Nevertheless, it is evident that in some cases modelling for various economic problems requires multivariate utility functions for which additivity on single components is not admissible: in the framework of bivariate utility functions two different approaches have been proposed in order to define the concept of bivariate risk

* This research was partially supported by MIUR.

aversion, namely of risk aversion to both risks together when a bivariate random variable (i.e. a vector) is considered.

De Finetti ([2]) and, independently, Richard ([15]) set the definition of bivariate risk aversion by considering the two lotteries

$$\mathcal{L}_1 = \begin{cases} (x_0, y_0), \text{ with probability } 0.5; \\ (x_1, y_1), \text{ with probability } 0.5 \end{cases}$$

$$\mathcal{L}_2 = \begin{cases} (x_0, y_1), \text{ with probability } 0.5; \\ (x_1, y_0), \text{ with probability } 0.5 \end{cases}$$

where $x_0 \leq x_1$ and $y_0 \leq y_1$.

Following their proposal a Decision Maker exhibits bivariate risk aversion if she prefers $\mathcal{L}_2$ to $\mathcal{L}_1$. As pointed out by the Authors, their definition of bivariate risk aversion does not require any condition on risk aversion in only one risk.

Extending the univariate concept of risk aversion, Kihlstrom and Mirman ([9]) based their notion of bivariate risk aversion on the comparison of a random vector with its expected value: more precisely, a Decision Maker is bivariate risk averse if she always prefers the expected value of a bi-dimensional random vector to the random vector itself. Differently from before, in this case a bivariate risk averse Decision Maker is necessarily risk averse to each particular risk.

Even if there is no a unanimous agreement on what is the right definition, one notion can be more adequate depending on the environment. Nevertheless, the proposal of Kihlstrom and Mirman exhibits a more appealing formulation: in fact stemming from the univariate notion of risk aversion their definition refers to the choice between a bivariate random vector and its expected value. Starting from the proposal of de Finetti and Richard in the present contribution the analysis is devoted to the study of a definition of bivariate risk aversion which is based on the choice of a Decision Maker between a bivariate random vector and another particular bivariate random vector: namely, its independent version. If she always prefers the independent version with the same univariate marginal distributions of a bivariate random vector which is positive dependent to the random vector itself then she exhibits bivariate risk aversion, that is she is *Bivariate Risk Averse* (BRA). In this definition the role played in the univariate case by the degenerate random variable $E(X)$ related to X is now played by the independent version $\mathbf{X}^{\perp}$ of $\mathbf{X}$ when $\mathbf{X}$ is positive dependent.

The interest in this new wording of the definition of de Finetti-Richard relies on the comparison between the two expected values $Eu(\mathbf{X}^{\perp})$ and $Eu(\mathbf{X})$.

As it is well-known, in Decision Theory and Actuarial Sciences it is almost usual to compare two random variables, that is risks, by stochastic orderings defined through inequalities on expectations of the random variables transformed by measurable functions of a cone $\mathcal{F}$. More precisely, given two random variables X and Y, an univariate integral stochastic ordering $\preceq_{\mathcal{F}}$ is defined as follows

$$X \preceq_{\mathcal{F}} Y \qquad \Longleftrightarrow \qquad E\phi(X) \leq E\phi(Y)$$

for all $\phi \in \mathcal{F} = \{\phi : \mathbb{R}_+ \to \mathbb{R}_+\}$, for which the expectations exist. By characterizing $\mathcal{F}$ some particular stochastic orderings may be obtained (such as stochastic dominance or stop-loss order). These stochastic order relations of integral form may be extended to cover also the case of random vectors (see, for instance, Denuit, Lefèvre and Mesfioui [3], in which the bivariate extension of a class of univariate orderings of convex-type is proposed).

In many situations individual risks are correlated since they are subject to the same claim generating mechanisms or are determined by the same environments. However, in traditional risk theory, individual risks are usually assumed to be independent for tractability. In recent years the study of the impact of dependence among risks has become a major topic, in particular in actuarial sciences. Several notions of dependence were introduced to model the fact that larger values of one of the component of a multivariate risk tend to be associated with larger values of the others. As it is well-known, different notions are equivalent in the bivariate case for risks with the same univariate marginal distribution ([23]) but this is no longer true for n-variate risks with $n \geq 3$ ([12]). Each of the proposed definitions captures different aspects of dependence, none of which with a main effectiveness on the others. In a way, under the term dependence many-sided notions are gathered together. The question rises when concordance and/or dependence between random vectors have to be described. In fact, there is not an unanimous opinion about their meaning and their appropriate definition. Anyway, in the judgement of many Authors there is the deep idea that the two definitions have to be distinguished. The present contribution is devoted to the analysis of the main characteristics a measure of concordance and/or dependence should have, as well as to the definition of some classes of measures of integral form. The framework is that of bivariate risks with the same marginal distributions. The context is that of association measures, that is of measures which are invariant under the order of indices, which are bounded by the measures of the upper and lower Fréchet bounds and are closed under weak convergence. All natural requirements are established by these desirable properties. One of our proposals is represented by the definition of conditions on which it is possible to distinguish between association measures of concordance and/or dependence. More precisely, a concordance measure is an association measure compatible with the concordance ordering, while dependence is ensured when an association measure is non-negative on PD random variables, non-positive on ND random variables. The attention is then focused on that association measures that are defined by integrals.

By considering the framework of Expected Utility Theory where each Decision Maker is assumed to have a utility function expressing her preferences and by referring to the context of bivariate risks with the same marginal distributions, the core of our proposal is represented by the definition of a form of bivariate risk attitude in such a way that it is possible to set a definition of bivariate stochastic order; the study of some plausible relations with some other stochastic orderings naturally follows.

The paper is organized as follows. In Section 2 we first recall some basic definitions for studying the problem of bivariate risk attitude and then we propose a notion of bivariate risk attitude. Next a characterization of this notion involving the utility function is deduced. On this result it is possible to set the definition of a stochastic order relation which is strongly related to the well-known submodular order, to the increasing submodular order and to the concordance order. Section 3 presents the main results related to the problem of measuring bivariate concordance and dependence.The following section presents the notion of concordance increasing transformation (CIT transformation): in this way it is possible to give a formal proof of the intuitive idea that "being more bivariate risk" is equivalent to being the limit of a sequence of bivariate random vectors which are increasing in concordance. Finally, section 5 is devoted to some conclusive observations.

2 Comparing bivariate risk attitude: notations and preliminary results

Some notations, abbreviations and conventions used throughout the paper are the following. Hereafter, we will only deal with non-negative random variables. $F^{\mathbf{X}}$ denotes the bi-dimensional cumulative distribution function (cdf) of the random vector $\mathbf{X} = (X_1, X_2)$, where $F^{\mathbf{X}}(\mathbf{t}) := P(\mathbf{X} \leq \mathbf{t}) = P(X_1 \leq t_1, X_2 \leq t_2)$, with $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$. The survival function corresponding to $F^{\mathbf{X}}$ is denoted by $S^{\mathbf{X}}$ and $S^{\mathbf{X}}(\mathbf{t}) := P(\mathbf{X} > \mathbf{t}) = P(X_1 > t_1, X_2 > t_2)$. For a 2-variate cdf $F^{\mathbf{X}}$ it is possible to write the marginal distribution and the marginal survival function of $\mathbf{X}$ such as follows:

$$F_i^{\mathbf{X}}(t_i) := P(X_i \leq t_i)$$
$$S_i^{\mathbf{X}}(t_i) := P(X_i > t_i)$$

where $i = 1, 2$. $\mathcal{R}$ denotes the *Fréchet space* given the margins, that is $\mathcal{R}(F_1, F_2)$ is the class of all the bivariate distributions with the given margins F_1, F_2. The lower Fréchet bound $\underline{\mathbf{X}}$ of $\mathbf{X}$ is defined by $F^{\underline{\mathbf{X}}}(\mathbf{t}) := \max\{F_1(t_1) + F_2(t_2) - 1, 0\}$ and the upper Fréchet bound of $\mathbf{X}$, $\overline{\mathbf{X}}$, is defined by $F^{\overline{\mathbf{X}}}(\mathbf{t}) := \min_i\{F_i(t_i)\}$, where $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$.

Definition 1 *Let $\mathbf{X}, \mathbf{Y}$ be bivariate random variables. Then*

a) $\mathbf{X} \preceq_{uo} \mathbf{Y}$ *if* $S^{\mathbf{X}}(\mathbf{t}) \leq S^{\mathbf{Y}}(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbb{R}^2$
b) $\mathbf{X} \preceq_{lo} \mathbf{Y}$ *if* $F^{\mathbf{X}}(\mathbf{t}) \geq F^{\mathbf{Y}}(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbb{R}^2$
c) $\mathbf{X} \preceq_{sm} \mathbf{Y}$ *if* $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ *for all supermodular functions* f *such that the expectations exist*
d) $\mathbf{X} \preceq_{c} \mathbf{Y}$ *if* $F^{\mathbf{X}}(\mathbf{t}) \leq F^{\mathbf{Y}}(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbb{R}^2$.

Clearly, indicator functions associated to upper orthants $[\mathbf{t}, +\infty) = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^2 \wedge x_1 \geq t_1, x_2 \geq t_2\}$ or lower orthants $(-\infty, \mathbf{t}] = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^2 \wedge x_1 \leq t_1, x_2 \leq t_2\}$ for a given $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, respectively $f = \mathbf{1}_{[\mathbf{t},\infty)}$ and $f = \mathbf{1}_{(-\infty,\mathbf{t}]}$, are supermodular. So the following implications may be immediately deduced:

$$\mathbf{X} \preceq_{sm} \mathbf{Y} \quad \Rightarrow \quad \mathbf{X} \preceq_{uo} \mathbf{Y}$$
$$\mathbf{X} \preceq_{sm} \mathbf{Y} \quad \Rightarrow \quad \mathbf{Y} \preceq_{lo} \mathbf{X}.$$

The previous results may be extended when the random variables have the same univariate marginal distributions ([23]).

Theorem 1 *Let* $\mathbf{X}, \mathbf{Y}$ *be bivariate random variables, where* $\mathbf{X}, \mathbf{Y} \in \mathcal{R}(F_1, F_2)$*. Then*

$$\mathbf{X} \preceq_{uo} \mathbf{Y} \quad \Leftrightarrow \quad \mathbf{Y} \preceq_{lo} \mathbf{X} \quad \Leftrightarrow \quad \mathbf{X} \preceq_{sm} \mathbf{Y} \quad \Leftrightarrow \quad \mathbf{X} \preceq_{c} \mathbf{Y}.$$

This result is no longer true when multivariate random variables are considered with $n \geq 3$ ([12]).

Definition 2 *Let* $\mathbf{X}$ *be a bivariate random variable. Then*

a) $\mathbf{X}$ *is PD (Positive Dependent) if and only if for all* $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$*,*

$$P(X_1 > t_1, X_2 > t_2) \geq P(X_1 > t_1)P(X_2 > t_2);$$

b) $\mathbf{X}$ *is ND (Negative Dependent) if and only if for all* $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$*,*

$$P(X_1 > t_1, X_2 > t_2) \leq P(X_1 > t_1)P(X_2 > t_2).$$

If $\mathbf{X} \in \mathcal{R}$, $\mathbf{X}^{\perp} = (X_1^{\perp}, X_2^{\perp})$ is the independent version of $\mathbf{X}$ in $\mathcal{R}$, that is $X_1^{\perp}$, $X_2^{\perp}$ are independent random variables and X_i, $X_i^{\perp}$ $(i = 1, 2)$ are identically distributed.

Definition 3 *A Decision Maker with bivariate utility function* u *is*

a) *Bivariate Risk Averse (BRA) if and only if for all* $\mathbf{X}$ *which is PD she prefers* $\mathbf{X}^{\perp}$ *to* $\mathbf{X}$*, that is*

$$Eu(\mathbf{X}) \leq Eu(\mathbf{X}^{\perp});$$

b) *Bivariate Risk Propense (BRP) if and only if for all* $\mathbf{X}$ *which is PD she prefers* $\mathbf{X}$ *to* $\mathbf{X}^{\perp}$*, that is*

$$Eu(\mathbf{X}) \geq Eu(\mathbf{X}^{\perp}).$$

As it is well-known, in the case of random variables the classic notion of risk aversion (propension) is equivalent to concavity (convexity) of the utility function: in the case of bivariate risk aversion for random vectors an analogous result may be stated too.

Theorem 2 *A Decision Maker with bivariate utility function* u *is BRA (resp. BRP) if and only if* u *is submodular (resp. supermodular), that is if and only if* u *satisfies the following inequality*

$$u(\mathbf{x} \vee \mathbf{y}) + u(\mathbf{x} \wedge \mathbf{y}) \leq u(\mathbf{x}) + u(\mathbf{y})$$

(resp. $u(\mathbf{x} \vee \mathbf{y}) + u(\mathbf{x} \wedge \mathbf{y}) \geq u(\mathbf{x}) + u(\mathbf{y})$*) for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$*, where* $\mathbf{x} \vee \mathbf{y}$ *is the componentwise maximum of* $\mathbf{x}$ *and* $\mathbf{y}$*,* $\mathbf{x} \wedge \mathbf{y}$ *is the componentwise minimum of* $\mathbf{x}$ *and* $\mathbf{y}$*.*

Proof

Let u be a submodular function: then the statement directly follows (see [23]) by $\mathbf{X}^{\perp} \preceq_c \mathbf{X}$, where $\preceq_c$ denotes the concordance order. Conversely, let the Decision-Maker be bivariate risk averse. Then the following lotteries $\mathbf{X}$ and $\mathbf{Y}$

$$\mathbf{X} = \begin{cases} (x_0, y_0), \text{ with probability } 0.25; \\ (x_0, y_1), \text{ with probability } 0.25; \\ (x_1, y_0), \text{ with probability } 0.25; \\ (x_1, y_1), \text{ with probability } 0.25 \end{cases}$$

and

$$\mathbf{Y} = \begin{cases} (x_0, y_0), \text{ with probability } 0.5; \\ (x_1, y_1), \text{ with probability } 0.5 \end{cases}$$

with $x_0 \leq x_1$ and $y_0 \leq y_1$, are ordered, that is

$$\frac{1}{4}\left(u(x_0, y_0) + u(x_0, y_1) + u(x_1, y_0) + u(x_1, y_1)\right) \geq \frac{1}{2}\left(u(x_0, y_0) + u(x_1, y_1)\right).$$

By subtracting $\frac{1}{2}\left(u(x_0, y_0) + u(x_1, y_1)\right)$ to each argument the inequality characterizing submodular functions follows.

Analogously for the case of BRP Decision Maker and supermodular utility function.

∎

Let us remark that the definition of bivariate risk aversion which has been suggested by de Finetti and proposed by Richard is based on the preference between the following two lotteries

$$\mathcal{L}_1 = \begin{cases} (x_0, y_0), \text{ with probability } 0.5; \\ (x_1, y_1), \text{ with probability } 0.5 \end{cases}$$

$$\mathcal{L}_2 = \begin{cases} (x_0, y_1), \text{ with probability } 0.5; \\ (x_1, y_0), \text{ with probability } 0.5 \end{cases}$$

where $x_0 \leq x_1$ and $y_0 \leq y_1$.

Following their proposal a Decision Maker exhibits bivariate risk aversion if she prefers $\mathcal{L}_2$ to $\mathcal{L}_1$.

Given the formulation of the choice defining a bivariate risk averse Decision Maker and the characterization of the associated bivariate utility function, now it is possible to introduce a stochastic order relation which is based on the notion of Bivariate Risk Aversion. A bivariate random vector will be less dangerous of another in a bivariate sense if it will be preferred by any bivariate risk averse Decision Maker. Analogously, a bivariate random vector will be more dangerous of another in a bivariate sense if it will be preferred by any bivariate risk propense Decision Maker.

Definition 4 *Let $\mathbf{X}$, $\mathbf{Y}$ be bivariate random variables in $\mathcal{R}$. Then $\mathbf{Y}$ is less dangerous in a bivariate sense then $\mathbf{X}$ (write $\mathbf{Y} \preceq_{b-r} \mathbf{X}$) if and only if $Eu(\mathbf{X}) \leq Eu(\mathbf{Y})$ for every submodular function u (namely, $Eu(\mathbf{X}) \geq Eu(\mathbf{Y})$ for every supermodular function u).*

This order is strongly related to some well-known stochastic orderings: by recalling these links, it is possible to give equivalent characterizations of bivariate risk aversion in terms of submodular order, increasing submodular order and of so called concordance order.

Theorem 3 *Let* $\mathbf{X}$, $\mathbf{Y}$ *be bivariate random variables in* $\mathcal{R}$. *The following conditions are equivalent:*

i) $\mathbf{Y} \preceq_{b-r} \mathbf{X}$;
ii) $Eu(\mathbf{X}) \leq E(u(\mathbf{Y})$ *for every increasing submodular function* u;
iii) $F^{\mathbf{Y}}(\mathbf{t}) \leq F^{\mathbf{X}}(\mathbf{t})$ *for all* $\mathbf{t} \in \mathbb{R}^2$;
iv) $E[f_1(X_1)f_2(X_2)] \leq E[f_1(Y_1)f_2(Y_2)]$ *for all increasing functions* f_1, f_2.

Proof

Condition *i)* obviously implies *ii)*. Conversely, let condition *ii)* be true. Since every submodular function is limit of increasing submodular functions, then (see Theorem 3.4 in [13]) the validity of condition *i)* is ensured. If *iii)* condition is assumed to be true, then by [23] the condition *i)* is necessarily satisfied. By same arguments, the converse is also true. To prove that *i)* $\Leftrightarrow$ *iv)* we refer to [3].

■

As previously mentioned, this result is no longer true when multivariate random variables are considered with $n \geq 3$ (see [12] and [13]).

3 Measuring bivariate random variables

In the previous section the proposed characterization of the stochastic order relation $\preceq_{b-r}$ is strongly related to the very well-known stochastic ordering of concordance $\preceq_c$ (in fact, $\mathbf{Y} \preceq_c \mathbf{X}$ if $F^{\mathbf{Y}}(\mathbf{t}) \leq F^{\mathbf{X}}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^2$). In this way, the notion of more bivariate risky random vector is expressed in terms of more concordant random vector (i.e. the random components of the bivariate random vector are more concordant). Recently, the question about the definition of concordance (and/or dependence) between random vectors opened the study of other definitions. In fact, there is not an unanimous opinion about their meaning and their appropriate definition. Now, we want to devote our analysis to the study of the main characteristics a measure of concordance and/or dependence should have, as well as to the definition of some classes of measures of integral form. We will refer to the context of association measures, that is of measures which are invariant under the order of indices, which are bounded by the measures of the upper and lower Fréchet bounds and are closed under weak convergence. Finally we will define a set of conditions on which it will be possible to distinguish between association measures of concordance and/or dependence. It is generally accepted that the population versions of Spearman's rho and Kendall's tau are measures of concordance: this is confirmed in our approach by characterizing particular classes of concordance measures of integral form.

Joe ([8]) defined a set of axioms that a bivariate dependence ordering of distributions should have in order that higher in the ordering means more positive concordance. With the aim of covering also the case of multivariate distribution functions, these properties have been generalized. In this paper we propose a modified version of the original proposal of Scarsini ([19]) about measures of dependence. Moreover, because the nature of association can assume a variety of forms, different notions of “dependence” between random variables will be proposed.

Definition 5 *A numeric measure $\alpha : \mathcal{R} \to I\!R$ is a measure of association if it satisfies the following properties:*

- *P1. α is defined for every pair of random variables X_1 and X_2;*
- *P2. α is invariant under permutation: $\alpha(X_1, X_2) = \alpha(X_2, X_1)$;*
- *P3. $\alpha(\mathbf{X}) \leq \alpha(\overline{\mathbf{X}})$, for all $\mathbf{X} \in \mathcal{R}$;*
- *P4. $\alpha(\underline{\mathbf{X}}) \leq \alpha(\mathbf{X})$, for all $\mathbf{X} \in \mathcal{R}$;*
- *P5. if $\{\mathbf{X}\}_n$ is a sequence of bivariate random vectors converging in distribution to $\mathbf{X}$, then $\lim_{n\to\infty} \alpha(\mathbf{X}_n) = \alpha(\mathbf{X})$.*

Property P2 establishes the invariance of a measure to order of indices; P3 and P4 guarantee natural multivariate association measures involving upper and lower Fréchet bounds, while P5 states the closure of a multivariate preordering under weak convergence.

Definition 6 *A numeric measure δ is a dependence measure if it is an association measure and it satisfies the properties:*

- *P6. $\delta(\mathbf{X}^{\perp}) \leq \delta(\mathbf{X})$ if $\mathbf{X}$ is PD;*
- *P7. $\delta(\mathbf{X}^{\perp}) \geq \delta(\mathbf{X})$ if $\mathbf{X}$ is ND.*

Definition 7 *A numeric measure γ is a concordance measure if it is an association measure and it satisfies the property:*

- *P8. if $\mathbf{X} \preceq_c \mathbf{Y}$ then $\gamma(\mathbf{X}) \leq \gamma(\mathbf{Y})$.*

Here we address our attention to study the role played by a particular class of functions in the characterization of a concordance measure: the family of supermodular functions. This choice will enable us to determine two classes of concordance measures of integral form: in this way, population version of Spearman's rho and Kendall's tau will result to be concordance measures. It is worthwhile to point out that in our approach even if Spearman's rho and Kendall's tau will be characterized as elements of two different classes of concordance measures, both these families are included in the same set of concordance measures of integral form. In this way, our study differs from the previous proposal of Scarsini ([19]): in fact, besides the different definition of concordance and the different framework, Spearman's rho belongs to the particular class of measures of concordance there defined as

$$\int_{\mathbf{I}^2} k\psi(u - 1/2)\psi(v - 1/2)\, dC^{\mathbf{X}}(u, v),$$

where $k = \left(\int_{\mathbf{I}} \psi^2(u-1/2)\,du\right)^{-1}$, ψ is a bounded odd function on $[-1/2, 1/2]$ and $\mathbf{I}^2 = [0,1] \times [0,1]$ denotes the unit square. This belonging is no longer true for Kendall's tau: as a consequence, the Author proposed a direct proof that it is a concordance measure.

Theorem 4 *Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be a right-continuous increasing bounded function. Then f is supermodular if and only if for every bounded positive Borel measures μ_1, μ_2 on $\mathbb{R}^2_+$, such that $\mu_1 \leq \mu_2$ and the integrals exist, the following inequality is satisfied*

$$\int_{\mathbb{R}^2_+} f d\mu_1 \leq \int_{\mathbb{R}^2_+} f d\mu_2.$$

Proof

Since f is a right-continuous increasing bounded function, then ([24]; [20])

$$f(x,y) = \int_{(x,y)}^{\infty} d\lambda(\mathbf{t}) + \int_{x}^{\infty} d\lambda_1(t_1) + \int_{y}^{\infty} d\lambda_2(t_2) + k$$

where λ is a positive measure on $\mathbb{R}^2_+$, λ_i $(i = 1,2)$ is a positive measure on $\mathbb{R}_+$, $k \in \mathbb{R}$. First, by Fubini's theorem,

$$\int_{\mathbb{R}^2_+} \int_{(x,y)}^{\infty} d\lambda(\mathbf{t}) d\mu(x,y) = \int_{\mathbb{R}^2_+} \int_{\mathbf{0}}^{(t_1,t_2)} d\mu(x,y) d\lambda(\mathbf{t}).$$

Analogous results may be set for each addend. Then, the hypothesis $\mu_1 \leq \mu_2$ yields

$$\int_{\mathbb{R}^2_+} f d\mu_1 \leq \int_{\mathbb{R}^2_+} f d\mu_2.$$

Conversely, let μ_1, $\mu_2 : \mathbb{R}^2_+ \to \mathbb{R}_+$ be positive functions defined as follows:

$$\mu_1(x,y) = F^{\mathbf{X}}(x,y),$$

$$\mu_2(x,y) = F^{\mathbf{Y}}(x,y)$$

where $\mathbf{X}$ and $\mathbf{Y}$ are the discrete random vectors

$$\mathbf{X} = \begin{cases} (x_1, y_2) & p = 1/2, \\ (x_2, y_1) & p = 1/2 \end{cases},$$

$$\mathbf{Y} = \begin{cases} (x_1, y_1) & p = 1/2, \\ (x_2, y_2) & p = 1/2 \end{cases}$$

and $x_1 \leq x_2$, $y_1 \leq y_2$. Clearly, $\mu_1 \leq \mu_2$. By hypothesis

$$\int_{\mathbb{R}^2_+} f d\mu_1 \leq \int_{\mathbb{R}^2_+} f d\mu_2$$

for every positive Borel measures on $\mathbb{R}^2_+$; hence f is supermodular.

∎

The following theorem establishes a characterization of supermodular functions via the Fréchet condition on $\mathbf{X}$ and $\mathbf{Y}$. In this way, the study of association is related to the analysis of the Fréchet class of bivariate distributions with fixed marginals, which in turn is fundamental in the study of positive dependence.

Theorem 5 *Let $f : \mathbf{I}^2 \to \mathbb{R}$ be a right-continuous increasing bounded function. Then f is supermodular if and only if for every continuous $\mathbf{X}, \mathbf{Y} \in \mathcal{R}$,*

$$\mathbf{X} \preceq_c \mathbf{Y} \Rightarrow \int_{\mathbb{R}^2_+} f(F_1^{\mathbf{X}}(t_1), F_2^{\mathbf{X}}(t_2)) dF^{\mathbf{X}}(\mathbf{t}) \leq \int_{\mathbb{R}^2_+} f(F_1^{\mathbf{Y}}(t_1), F_2^{\mathbf{Y}}(t_2)) dF^{\mathbf{Y}}(\mathbf{t}).$$

Proof

By assuming $\mathbf{X}$ and $\mathbf{Y}$ in the same space $\mathcal{R}$, the assertion

$$\int_{\mathbb{R}^2_+} f(F_1^{\mathbf{X}}(t_1), F_2^{\mathbf{X}}(t_2)) dF^{\mathbf{X}}(\mathbf{t}) \leq \int_{\mathbb{R}^2_+} f(F_1^{\mathbf{Y}}(t_1), F_2^{\mathbf{Y}}(t_2)) dF^{\mathbf{Y}}(\mathbf{t})$$

is equivalent to

$$\int_{\mathbf{I}^2} f(u,v) dC_{\mathbf{X}}(u,v) \leq \int_{\mathbf{I}^2} f(u,v) dC_{\mathbf{Y}}(u,v)$$

where $u = F_1(t_1)$ and $v = F_2(t_2)$; $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$ are the copulas associated to $F^{\mathbf{X}}$ and $F^{\mathbf{Y}}$ respectively, that is

$$C_{\mathbf{X}}(u,v) = F^{\mathbf{X}}((F_1^{\mathbf{X}})^{-1}(u), (F_2^{\mathbf{X}})^{-1}(v))$$
$$C_{\mathbf{Y}}(u,v) = F^{\mathbf{Y}}((F_1^{\mathbf{Y}})^{-1}(u), (F_2^{\mathbf{Y}})^{-1}(v))$$

where $F^{(-1)}$ is the quasi-inverse of the distribution F. The conclusion now follows from Theorem 4.

∎

The next result shows that, given a symmetric right-continuous increasing bounded supermodular function f, it is possible to define a concordance measure which is related to it by an integral transform.

Corollary 1 *Let $f : \mathbf{I}^2 \to \mathbb{R}$ be a symmetric right-continuous increasing bounded supermodular function. Then $\int_{\mathbb{R}^2_+} f(F_1^{\mathbf{X}}, F_2^{\mathbf{X}}) dF^{\mathbf{X}}$ is a measure of concordance.*

Proof

It is immediate to see from definition that the first four properties characterizing a concordance measure are verified. Property $P5$ is verified for the Theorem of Helly-Bray.

∎

Corollary 2 *The population version of Spearman's rho is a measure of concordance on the class of continuous distributions of $\mathcal{R}$.*

Proof

The result easily follows by setting $f(x,y) = xy$. In fact, as it is well-known, if $C^{\mathbf{X}}$ is the copula of the continuous bivariate random vector $\mathbf{X}$,

$$\rho_{(X_1,X_2)} = 12 \int_{\mathbf{I}^2} uv \, dC^{\mathbf{X}}(u,v) - 3,$$

expression which is equivalent to the following one

$$\rho = 12 \int_{\mathbb{R}^2_+} F_1^{\mathbf{X}}(t_1) F_2^{\mathbf{X}}(t_2) \, dF^{\mathbf{X}}(t_1,t_2) - 3.$$

∎

The attention is now directed to another class of functions on which it is possible to define an ordering of integral form which is compatible with the concordance one. On this result is based the characterization of a new class of concordance measures.

Theorem 6 *Let $f : \mathbf{I} \to \mathbb{R}$ be an increasing convex function, and let $\mathbf{X}, \mathbf{Y}$ be in $\mathcal{R}$. Then*

$$\mathbf{X} \preceq_c \mathbf{Y} \Rightarrow \int_{\mathbb{R}^2_+} f(F^{\mathbf{X}}(\mathbf{t})) dF^{\mathbf{X}}(\mathbf{t}) \leq \int_{\mathbb{R}^2_+} f(F^{\mathbf{Y}}(\mathbf{t})) dF^{\mathbf{Y}}(\mathbf{t}).$$

Proof

Since $F^{\mathbf{X}}$ is a monotone function, $f(F^{\mathbf{X}})$ is a supermodular monotone function (see [11]. Then by Theorem 4

$$\int_{\mathbb{R}^2_+} f(F^{\mathbf{X}}(\mathbf{t})) dF^{\mathbf{X}}(\mathbf{t}) \leq \int_{\mathbb{R}^2_+} f(F^{\mathbf{X}}(\mathbf{t})) dF^{\mathbf{Y}}(\mathbf{t})$$

and since $\mathbf{X} \preceq_c \mathbf{Y}$,

$$\int_{\mathbb{R}^2_+} f(F^{\mathbf{X}}(\mathbf{t})) dF^{\mathbf{Y}}(\mathbf{t}) \leq \int_{\mathbb{R}^2_+} f(F^{\mathbf{Y}}(\mathbf{t})) dF^{\mathbf{Y}}(\mathbf{t}).$$

∎

Corollary 3 *Let $f : \mathbf{I} \to \mathbb{R}$ be an increasing, convex function, and let $\mathbf{X}, \mathbf{Y}$ be in $\mathcal{R}$. Then $\int_{\mathbb{R}^2_+} f(F^{\mathbf{X}}) dF^{\mathbf{X}}$ is a measure of concordance.*

In a similar chain of results as those related to Corollary 2 on Spearman's rho, the population version of Kendall's tau results to be a concordance measure.

Corollary 4 *The population version of Kendall's tau is a measure of concordance on the class of continuous distributions of $\mathcal{R}$.*

Proof

The result is a direct consequence of the previous statement by setting $f(x) = x$. In fact,

$$\tau_{(X_1,X_2)} = 4 \int_{\mathbb{R}^2} F^{\mathbf{X}}(\mathbf{t})\, dF^{\mathbf{X}}(\mathbf{t}) \; - 1.$$

■

4 Comparing bivariate risk attitude through *CIT* transformations

The equivalent characterizations of bivariate riskiness through some bivariate stochastic orderings which has been proposed in the previous section now open the study of transformations of bivariate vectors leading to more dangerous bivariate vectors. If a more concordant vector is more risky, then the attention may be addressed to the analysis of transformations that increase the concordance, namely the bivariate riskiness.

The following definition introduces the notion of *Concordance Increasing Transformation*, written *CIT*.

Definition 8 *Let $\mathbf{X}$, $\mathbf{Y}$ be bivariate random variables in $\mathcal{R}$. Then $\mathbf{Y}$ is a CIT of $\mathbf{X}$ if and only if there exist two subsets of $\mathbb{R}^2$, say $\mathcal{A}$, $\mathcal{B}$, such that from*

$$\mathbf{X}|_{\mathcal{A}} = \begin{cases} \mathbf{a}, & \text{with probability } p_1; \\ \mathbf{b}, & \text{with probability } p_2; \\ \mathbf{a} \wedge \mathbf{b}, & \text{with probability } p_3; \\ \mathbf{a} \vee \mathbf{b}, & \text{with probability } p_4 \end{cases}$$

it follows that

$$\mathbf{Y}|_{\mathcal{B}} = \begin{cases} \mathbf{a}, & \text{with probability } p_1 - \epsilon_1; \\ \mathbf{b}, & \text{with probability } p_2 - \epsilon_2; \\ \mathbf{a} \wedge \mathbf{b}, & \text{with probability } p_3 + \epsilon_1; \\ \mathbf{a} \vee \mathbf{b}, & \text{with probability } p_4 + \epsilon_2 \end{cases}$$

where $0 \leq \epsilon_i \leq p_i$.

The next theorem sets an equivalence result between the notion of riskier bivariate random vector and the existence of a sequence of bivariate random vectors each of them being an increasing concordance transformation of the previous one.

Theorem 7 $\mathbf{Y} \preceq_{b-r} \mathbf{X}$ *if and only if there exist sequences $\{\mathbf{Y}_n\}$ and $\{\mathbf{X}_n\}$ of bivariate random variables such that $\mathbf{Y}_n \to \mathbf{Y}$ and $\mathbf{X}_n \to \mathbf{X}$ in distribution and for any $\mathbf{X}_i$ and $\mathbf{Y}_i$ there exists a finite sequence $\mathbf{Z}_{i0} \equiv \mathbf{Y}_i, \mathbf{Z}_{i1}, \cdots, \mathbf{Z}_{ik(i)} \equiv \mathbf{X}_i$ such that $\mathbf{Z}_{i(j+1)}$ differs from $\mathbf{Z}_{ij}$ $(0 \leq j \leq k(i))$ by a CIT.*

Proof

If $\mathbf{Y} \preceq_{b-r} \mathbf{X}$ then it follows that by Theorem 2 $\mathbf{Y} \preceq_c \mathbf{X}$, that is $F^{\mathbf{Y}}(\mathbf{t}) \leq F^{\mathbf{X}}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^2$. By Theorem 2 of [5] there exist discrete sequences $\{\mathbf{Y}_n\}$ and and $\{\mathbf{X}_n\}$ such that $\mathbf{Y}_n \to \mathbf{Y}$ and $\mathbf{X}_n \to \mathbf{X}$ pointwise. Moreover, any $\mathbf{Y}_i \preceq_c \mathbf{X}_i$ for any i. Then by Theorem 1 of [5] any $\mathbf{X}_i$ is obtained from $\mathbf{Y}_i$ by a finite sequence of CIT's.

The converse directly follows by Theorem 2 and by closure property of concordance order with respect to limit in distribution.

∎

5 Concluding remarks

In Decision Theory the concept of risk aversion has been studied from different angles. The reason for studying risk attitude is the well-known fact that if a Decision Maker prefers the expected value of a risk to the risk itself then the risk premium is positive and the utility function is concave. In this framework by paying the relative risk premium any risk can always be totally eliminated: in other words, full insurance is always possible. Subsequently, partial insurance has been studied in literature and the reference framework was usually that of additive risks with utility functions depending only on one argument. Nevertheless, in many situations is it often required to consider multivariate utility functions for which additivity on single components is not admissible: in the framework of bivariate utility functions two different approaches have been proposed in order to define the concept of bivariate risk aversion, namely of risk aversion to both risks together when a bivariate random variable (i.e. a vector) is considered. Dealing with the proposal of de Finetti-Richard in this note we propose a different formulation of their proposal in order to display a link with the definition of Arrow-Pratt. Some equivalent characterizations of the related bivariate stochastic ordering are presented: they are related to one notion of concordance. This is why we moved our attention to the study of different notions of measures of concordance/dependence, in a unified framework for studying dependence association measures and concordance association measures for bivariate distributions.

Our proposal may be extended to cover also the case of higher order bivariate risk attitude. In fact, note that starting from the two lotteries

$$\mathcal{L}_1^1(x_0, y_0) = \mathcal{L}_1 = \begin{cases} (x_0, y_0), \text{ with probability } 0.5; \\ (x_1, y_1), \text{ with probability } 0.5 \end{cases}$$

$$\mathcal{L}_2^1(x_0, y_0) = \mathcal{L}_2 = \begin{cases} (x_0, y_1), \text{ with probability } 0.5; \\ (x_1, y_0), \text{ with probability } 0.5 \end{cases}$$

where $x_0 \leq x_1$ and $y_0 \leq y_1$, it is possible to define the following iterated lotteries:

$$\mathcal{L}_1^k(x_0, y_0) = \begin{cases} \mathcal{L}_1^{k-1}(x_0, y_0), \text{ with probability } 0.25; \\ \mathcal{L}_2^{k-1}(x_1, y_0), \text{ with probability } 0.25; \\ \mathcal{L}_2^{k-1}(x_0, y_1), \text{ with probability } 0.25; \\ \mathcal{L}_1^{k-1}(x_1, y_1), \text{ with probability } 0.25 \end{cases}$$

$$\mathcal{L}_2^k(x_0, y_0) = \begin{cases} \mathcal{L}_2^{k-1}(x_0, y_0), \text{ with probability } 0.25; \\ \mathcal{L}_1^{k-1}(x_1, y_0), \text{ with probability } 0.25; \\ \mathcal{L}_1^{k-1}(x_0, y_1), \text{ with probability } 0.25; \\ \mathcal{L}_2^{k-1}(x_1, y_1), \text{ with probability } 0.25. \end{cases}$$

Moreover, starting from the non equivalence of some well-known definitions of concordance and/or dependence when $n = 3$, the interest for different notions of multivariate aversion naturally follows (see, for example, Hu, Müller and Scarsini [6]).

References

1. Arrow, K.J.: Essays in the Theory of Risk Bearing. North Holland, New York (1974).
2. De Finetti, B.: Sulla preferibilità . Giornale degli Economisti e Annali di Economia, 11 (1952) 685–709.
3. Denuit, M., LefèVre, C., Mesfioui, M.: A class of bivariate stochastic orderings, with applications in actuarial sciences. Insurance: Mathematics and Economics, 24 (1999) 31–50.
4. Denuit, M., LefèVre, C., Scarsini, M.: On S-convexity and risk aversion. Theory and Decision, 50 (2001) 239–248.
5. Epstein, L.G., Tanny, S.M.: Increasing generalized correlation: a definition and some economic consequences. Canadian Journal of Economics, 13 (1980) 16–34.
6. Hu, T. Müller, A., Scarsini, M.: Some counterexamples in positive dependence. Journal of Statistical Planning and Inference, 124 (2004)153–158.
7. Joe, H.: Multivariate Concordance. Journal of Multivariate Analysis, 35 (1990) 12–30.
8. Joe, H.: Multivariate Models and Dependence Concepts. Chapman & Hall, London (1997).
9. Kihlstrom, R.E., Mirman, L.J.: Risk aversion with many commodities. Journal of Economic Theory, 8 (1974) 361–388.
10. Landsberger, M., Meilijson, I.: A tale of two tails: an alternative characterization of comparative risk. Journal of Risk and Uncertainty, 3 (1990) 65–82.
11. Marshall, A.W., Olkin I.: Inequalities: Theory of Majorization and Its Applications. Academic Press, New York (1979).
12. Müller, A.: Stop-loss order for portfolios of dependent risks. Insurance: Mathematics and Economics, 21 (1997) 219–223.
13. Müller, A., Scarsini M.: Some remarks on the Supermodular Order. Journal of Multivariate Analysis, 73 (2000) 107–119.
14. Pratt, J.W.: Risk aversion in the small and in the large. Econometrica, 32 (1964) 122–136.

15. Richard, S.F.: Multivariate risk aversion, utility independence and separable utility functions. Management Science, 22 (1975) 12–21.
16. Rothshild, M., Stiglitz, J.: Increasing risk: I. A definition. Journal of Economic Theory, 2(4) (1970) 225–243.
17. Rothshild, M., Stiglitz, J.: Increasing risk: II. Its economic consequences. Journal of Economic Theory, 3(1) (1971) 66–84.
18. Rothshild, M., Stiglitz, J.: Addendum to: "Increasing risk. I. A definition". Journal of Economic Theory, 5(2) (1972) 306.
19. Scarsini, M.: On measures of concordance. Stochastica, 8 (1984) 201–218.
20. Scarsini, M.: Stochastic Dominance with Pair-Wise Risk Aversion. Journal of Mathematical Economics, 14 (1985) 187–201.
21. Scarsini, M.: Dominance conditions for multivariate utility functions. Management Science, 34 (1988) 454–460.
22. Scarsini, M.: On risk aversion with two risks. Journal of Mathematical Economics, 31 (1999) 239–250.
23. Tchen, A.H.: Inequalities for distributions with given marginales. Annals of Probability, 8 (1980) 814–827.
24. von Neumann, J.: Functional Operators. Vol.I: Measures and Integrals. Princeton University Press, Princeton, N.J. (1950).